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Auxiliary Ghost Fields in Statistical Dynamics

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The path integral formulation of statistical dynamics involves a functional determinant whose role within the theory has remained somewhat unclear. This has occasionally led to incorrect generalizations of the formalism to the case of multiplicative random forces. We present a hidden symmetry of the theory and show how it can be used to clarify these issues. Important further applications are also pointed out.

KEY WORDS: Classical statistical dynamics; functional integral formalism; Becchi–Rouet–Stora symmetries; Ward identities; stochastic differential equations.

1. INTRODUCTION

Classical statistical dynamics is, roughly speaking, a theory of classical (usually nonlinear) systems, in which randomness is introduced by means of stochastic external driving terms and/or random initial conditions. A satisfactory formulation was lacking until Martin *et al.*[1] proposed an operator formalism, patterned after the Schwinger algorithm in quantum field theory (QFT), that allowed the implementation of a systematic approximation procedure. Several authors quickly realized that the theory admitted a functional integral formulation[2] which closely parallels the modern version of QFT (for an excellent review of both the operator and functional versions of the theory, see Jensen[3]). Although it is undeniable that crucial differences between these two approaches exist, in this paper we wish to pursue a further analogy that will permit the application of a powerful tool widely used in gauge theories, namely BRS invariances.[4] The emergence of such a symmetry is not difficult to understand. When the path integral formula for the generating functional is written down, a deter-

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minant appears as a (partial) reflection of the fact that we are dealing with a constrained theory, i.e., the fields must satisfy the stochastic differential equation that provided the basic starting point. This determinant can then be turned into an integral over anticommuting fields (ghosts) in a standard fashion, and one is ready to start searching for transformations connecting commuting and anticommuting fields which leave the "action" invariant. The ultimate reason for the existence of this (BRS) symmetry has been clearly pointed out by Zinn-Justin[5]: it is nothing more than the expression, within the ghost formalism, of the constraint mentioned above. Since this is an invariance of the full generating functional, it implies relationships between the exact Green's (or correlation) functions without having to resort to perturbation theory. This observation forms the backbone of the present paper, and we will use it to show, in a nonperturbative setting, how it effectively reduces the number of Green's functions to be computed as well as the crucial role played by the determinant.

The organization of the paper is as follows. In Section 2 we give a brief account of the usual treatment of the determinant, and explain why we regard it as unsatisfactory. The theory is shown to be BRS invariant in Section 3, and in Section 4 we proceed to practical applications of this result. In Section 5 we discuss the elimination of the conjugate field in the specific case of a white noise force. Finally, Section 6 is devoted to a remark concerning the problem of renormalizability and conclusions.

2. REVIEW OF THE STANDARD APPROACH

We consider a field ϕ that satisfies the equation

$$\partial_t \phi + F[\phi] = f \tag{1}$$

where $F[\phi]$ contains no time derivatives, and the random force f has a Gaussian distribution

$$\langle f(x)\, f(y) \rangle = K(x, y) \tag{2}$$

K is a given function, and x stands for $(\mathbf{x}, t)$.

To compute correlations one would have to solve Eq. (1) for ϕ as a functional of f and then average over f. Alternatively, one can proceed via the generating functional[2]

$$Z[J, L] = \int D\phi\, D\hat{\phi} \det\left(\partial_t + \frac{\delta F}{\delta \phi}\right) \exp\left\{ i \int \hat{\phi}(\partial_t \phi + F)\, d^4x \right.$$
$$\left. - \frac{1}{2} \iint \hat{\phi}(x)\, K(x, y)\, \hat{\phi}(y)\, d^4x\, d^4y + \int J\phi\, d^4x + \int L\hat{\phi}\, d^4x \right\} \tag{3}$$

and use functional derivatives with respect to J and/or L.

A slightly modified version of the usual argument would then run as follows. Assume that F has a term linear in ϕ and set

$$F = \hat{H}\phi + F_2 \tag{4}$$

where F_2 contains the nonlinear pieces. For example, $\hat{H} = -\nu\nabla^2$ for the Navier–Stokes equation. If G denotes the retarded Green's function for $\partial_t + \hat{H}$, then, in matrix notation,

$$\begin{aligned}\det\left(\partial_t + \frac{\delta F}{\delta\phi}\right) &= \det(\partial_t + \hat{H}) \det\left(1 + G\frac{\delta F_2}{\delta\phi}\right)\\ &= \det(\partial_t + \hat{H}) \exp \operatorname{Tr} \ln\left(1 + G\frac{\delta F_2}{\delta\phi}\right)\end{aligned} \tag{5}$$

Due to the retarded nature of G, only the first term survives in the expansion of the logarithm. Taking the trace then yields

$$\det\left(\partial_t + \frac{\delta F}{\delta\phi}\right) = \det(\partial_t + \hat{H}) \exp\left\{\theta(0) \int \frac{\delta F_2}{\delta\phi}(x)\, d^4x\right\} \tag{6}$$

The symbol $\delta F_2/\delta\phi(x)$ stands for the result of computing $\delta F_2(x)/\delta\phi(y)$, dropping the $\delta(t_x - t_y)$, and finally setting $\mathbf{x} = \mathbf{y}$. It is here that the purely formal nature of this method becomes apparent. The last step generates a $\delta^3(0)$, which should be properly regularized in order for the expression to make sense. This difficulty, which is a direct consequence of the fact that we are dealing with functional determinants instead of finite-dimensional ones, does not arise in the ghost formulation.

Even if one ignores this problem, as is usually done, one has to provide a definition of the quantity $\theta(0)$. The customary choices, $\theta(0) = 0$ and $\theta(0) = 1/2$, are directly related to the Itô and Stratonovich interpretations[6] of Eq. (1). The first option completely eliminates the determinant from the theory, a fact that immediately raises the question of the relationship between the resulting formulations. Fortunately, for non-multiplicative random forces, the answer turns out to be that they are equivalent, regardless of the value assigned to $\theta(0)$. To prove this, introduce a loop counting parameter a into the generating functional (see, e.g., ref. 7), and do a perturbation expansion in a. One finds that the term $\theta(0)\, \delta F_2/\delta\phi$ appearing in (6) gets multiplied by $(1 - a)$, where the first factor comes from the original determinant, and the second from the interaction term $i\hat{\phi}F_2$ in Eq. (3), a $\hat{\phi}\phi$ leg in the graph closing onto a loop (hence the factor a). Since a must be set equal to one in the end, we conclude that the determinant drops out irrespective of the value of $\theta(0)$.

Again, the need to assign a value to an otherwise undetermined quantity never arises when ghosts are used.

The above analysis is in agreement with known facts in the theory of stochastic differential equations.[6] There it is also well known, however, that the Itô and Stratonovich prescriptions are not equivalent in the case of multiplicative random forces, and therefore consistency demands that the corresponding path integral formulas, too, should predict physically inequivalent results. Recent experiments seem to favor the Stratonovich interpretation, at least in the few cases studied.[8] Whether this is true in general is unknown and reflects the lack of a theoretical basis to make a choice for systems involving strictly white noise. For a real noise with finite correlation time, the natural interpretation is that of Stratonovich, and one could then argue that the results reported in ref. 8 are to be expected if one regards white noise as a limiting case. In the following we restrict ourselves to nonmultiplicative forces.

To summarize, the usual treatment involves dealing with a divergent $[\delta^3(0)]$ and an undefined $[\theta(0)]$ expression. Moreover, the proof that this undefined expression is of no consequence to the physical observables can only be carried out in perturbation theory. The strong appeal of the ghost formulation lies in the fact that such ambiguous quantities never enter the theory. In addition, the presence of the ghosts reveals a symmetry which can be used, among other things, to show at the nonperturbative level the cancellation of the contribution coming from the determinant with that coming from a $\hat{\phi}\phi$ interaction term. Since this symmetry also has a bearing on the problem of renormalizability, we feel that ghosts are a powerful alternative to the conventional methods.

3. GHOSTS AND BRS INVARIANCE

In general, a BRS invariance is a reflection of a symmetry of the action with ghosts included. As such, it manifests itself as an identity satisfied by the generating functional, and therefore also as a tower of identities between Green's functions.

Ghosts were first introduced into statistical dynamics in ref. 9 (for the specific case of turbulence, see ref. 10) using a standard prescription from QFT that allows one to rewrite the determinant in Eq. (3) as a functional integral over anticommuting Grassmann fields (denoted here by η and $\bar{\eta}$). The generating functional takes the form

$$Z[J, L, \zeta, \bar{\zeta}] = \int D\phi\, D\hat{\phi}\, D\bar{\eta}\, D\eta \times \exp\left\{S[\phi, \hat{\phi}, \bar{\eta}, \eta] + \int (J\phi + L\hat{\phi} + \zeta\eta + \bar{\zeta}\bar{\eta})\, d^4x\right\} \tag{7}$$

where S is the "action":

$$S[\phi, \hat{\phi}, \bar{\eta}, \eta] = i \int \hat{\phi}(\partial_t \phi + F)\, d^4x - \frac{1}{2} \iint \hat{\phi}(x)\, K(x, y)\, \hat{\phi}(y)\, d^4x\, d^4y$$

$$- \iint \bar{\eta}(x) \left[\partial_t \delta^4(x-y) + \frac{\delta F(x)}{\delta \phi(y)} \right] \eta(y)\, d^4x\, d^4y \qquad (8)$$

Let us remark that the following discussion does not depend at all on the precise form of the left-hand side of Eq. (1). One may replace it by a general function of ϕ without affecting (11) below.

A general variation of S produces the expression (we suppress integrations for simplicity)

$$\delta S = i\, \delta\hat{\phi}(\partial_t \phi + F) + i\hat{\phi}\left(\partial_t + \frac{\delta F}{\delta \phi}\right) \delta\phi - \hat{\phi} K\, \delta\hat{\phi}$$

$$- \delta\bar{\eta}\left(\partial_t + \frac{\delta F}{\delta \phi}\right) \eta - \bar{\eta}\, \frac{\delta^2 F}{\delta^2 \phi}\, \delta\phi\, \eta - \bar{\eta}\left(\partial_t + \frac{\delta F}{\delta \phi}\right) \delta\eta$$

We note that there is only one term involving $\delta^2 F/\delta^2 \phi$. This can be eliminated if we put $\delta\phi = \varepsilon\eta$, with ε an infinitesimal anticommuting parameter, since $\delta^2 F/\delta^2 \phi$ is symmetric, whereas $\eta\eta$ is antisymmetric. The second term on the rhs then becomes $i\varepsilon\hat{\phi}(\partial_t + \delta F/\delta\phi)\eta$, which can clearly be canceled against the term containing $\delta\bar{\eta}$, if we set $\delta\bar{\eta} = i\varepsilon\hat{\phi}$. Finally, we observe that the only possible choice for the remaining variations is $\delta\hat{\phi} = 0$, and $\delta\eta = 0$, if we restrict ourselves to linear transformations. This shows that S is invariant under the transformation

$$\begin{aligned} \delta\hat{\phi} &= 0 \\ \delta\eta &= 0 \\ \delta\phi &= \varepsilon\eta \\ \delta\bar{\eta} &= i\varepsilon\hat{\phi} \end{aligned} \qquad (9)$$

One may also show from (9) that this is a nilpotent transformation, i.e., the second variation of the fields vanishes.

As mentioned in the introduction, this symmetry is just a consequence of the constrained nature of the theory, in which the determinant provides an invariant measure for ϕ. It is interesting to note that our system does not possess an anti-BRS symmetry[11] (that is, a symmetry relating $\delta\phi$ to the antighost $\bar{\eta}$ instead of to the ghost η) unless $F_2 = 0$, and $H = -H^+$, where H^+ is the adjoint of H. On the other hand, systems that are invariant under time reversal possess an anti-BRS provided F satisfies the integrability condition $\delta F(x)/\delta\phi(y) = \delta F(y)/\delta\phi(x)$. At this point, we remind the reader that we have imposed initial conditions on the fields and, consequently, one should regard the determinant as intrinsically retarded. This fact is responsible for the unfamiliar requirements on F_2 and H above, as well as for the absence of the supersymmetry discussed in ref. 9, which arises only if F is integrable, and all fields are assumed to be periodic. Gozzi[12] has shown that periodicity of the fields is equivalent to a half-retarded, half-advanced dynamics. Therefore the supersymmetry has no bearing on our problem, except possibly in the long-time limit, where the supersymmetric theory apparently produces the correct equilibrium distribution. Furthermore, even if the anti-BRS did exist, it would be the conjugate of the BRS, and thus $\delta\hat{\phi} = 0$ would remain true after combining the two. This is evidence that the supersymmetry can only be obtained from the BRS if one supplements it with a more general symmetry.

Since S is invariant under (9), we find that a change of variables in the generating functional affects only the source terms (the Jacobian is clearly equal to one)

$$Z[J, L, \zeta, \bar{\zeta}] = \int D\phi\, D\hat{\phi}\, D\eta\, D\bar{\eta} \times \exp(S + J\phi + L\hat{\phi} + \zeta\eta + \bar{\zeta}\bar{\eta} + J\,\delta\phi + \bar{\zeta}\,\delta\bar{\eta}) \qquad (10)$$

Replacing $\delta\phi$ and $\delta\bar{\eta}$ from (9) and expanding in ε gives us two terms; the first reproduces Z, and therefore the second must vanish, which leads finally to

$$\left(J\frac{\delta}{\delta\zeta} - i\bar{\zeta}\frac{\delta}{\delta L}\right) Z[J, L, \zeta, \bar{\zeta}] = 0 \qquad (11)$$

This Ward identity for Z plays a key role in what follows.

4. SOME CONSEQUENCES OF THE WARD IDENTIFY

Our first task in this section will be to show how (11) can be used to reduce the number of correlation functions to be computed. Consider two functions $C_1[\bar{\eta}, \eta, \hat{\phi}]$ and $C_2[\eta, \phi, \hat{\phi}]$. To compute their averages (which we denote by $\langle \cdot \rangle$) with $\hat{\phi}$ and η, respectively, we only need to replace fields by derivatives with respect to sources, apply the resulting operator to Z, and set the sources equal to zero. For C_1, this is the same as applying $iC_1[\delta/\delta\bar{\zeta}, \delta/\delta\zeta, \delta/\delta L]\,\delta/\delta\bar{\zeta}$ to (11) and setting the sources to zero. A similar procedure can be followed for C_2 to conclude that

$$\langle C_1[\bar{\eta}, \eta, \hat{\phi}]\hat{\phi}\rangle = 0 \qquad (12)$$

$$\langle C_2[\eta, \phi, \hat{\phi}]\eta\rangle = 0 \qquad (13)$$

Note that they imply, in particular, the vanishing of the exact $\bar{\eta}\hat{\phi}$, $\eta\hat{\phi}$, $\hat{\phi}\hat{\phi}$, $\eta\eta$, and $\eta\phi$ propagators. It is a simple excercise to prove that only $\phi\phi$, $\phi\hat{\phi}$, and $\bar{\eta}\eta$ have nonvanishing propagators, with

$$\langle \bar{\eta}\eta \rangle = i\langle \phi\hat{\phi} \rangle \tag{14}$$

also a consequence of (11).

Of course, the Ward identity has many other implications. Our main interest here is to derive from it an equation showing how the contribution of the determinant is canceled by other terms in S in a nonperturbative fashion. We start by assuming that F_2 in Eq. (2) can be expanded in a series in ϕ (this is not a serious restriction in practice, and it has to be assumed in the calculation of Section 2 also). Then, if we let

$$\hat{F}_{2x} \equiv F_2\left[\frac{\delta}{\delta J(x)}\right]$$

$$\hat{F}_{2xy} \equiv \left.\frac{\delta F_2[\phi(x)]}{\delta\phi(y)}\right|_{\phi \to J}$$

we can show that, for any function $A(J)$,

$$\hat{F}_{2x} J(y)\, A(J) = J(y)\, \hat{F}_{2x} A(J) + \hat{F}_{2xy} A(J) \tag{15}$$

Differentiating (11) with respect to $\bar{\zeta}(x)$ and setting all sources except J equal to 0, we find

$$\int J(z) < \bar{\eta}(x)\, \eta(z) > d^4z = i\langle \hat{\phi}(x) \rangle \tag{16}$$

Measurable quantities pick up contributions from the ghosts only through the interaction term $\bar{\eta}\, \delta F_2/\delta\phi\, \eta$, in which the arguments of $\bar{\eta}$ and η are integrated over. Acting on (16) with $\hat{F}_{2y}$ and using (15), we obtain, in the limit $J \to 0$,

$$\int \left\langle \bar{\eta}(x)\, \frac{\delta F_2[\phi(y)]}{\delta\phi(z)}\, \eta(z) \right\rangle d^4z = i\langle F_2[\phi(y)]\, \hat{\phi}(x) \rangle \tag{17}$$

One may now put $x = y$ and integrate; in our condensed notation this reads

$$\left\langle \bar{\eta}\, \frac{\delta F_2}{\delta\phi}\, \eta \right\rangle = i\langle F_2 \hat{\phi} \rangle \tag{18}$$

The right-hand side is precisely the $\phi\hat{\phi}$ interaction term, except for the sign. Hence (18) tells us that the sum of these contributions will vanish, as can be verified order by order in perturbation theory. Since the absence of $\phi\hat{\phi}$ loops is necessary to maintain causality, we can interpret (18) as stating that the determinant ensures that causality as well as the constraint (1) is obeyed exactly.

In agreement with the remark after Eq. (8), one may substitute F_2 in (18) by any observable $O[\phi]$, as long as it can be expressed as a series in ϕ. This establishes a general relationship between the average response function $\langle O[\phi]\hat{\phi}\rangle$ and the ghosts, of which (18) and (14) are particular examples.

5. THE CASE OF A WHITE NOISE STIRRING FORCE

We shall now apply the results of Sections 3 and 4 to a system forced by Gaussian white noise. For this case it has been shown that one can perform an integration over the conjugate field $\hat{\phi}$, thus eliminating it from the generating functional.[10] This is a significant departure from theories of statistical dynamics utilizing the conjugate field where it must be included in the calculation of the full set of correlation functions. While the physical significance of the $\hat{\phi}$ field by itself is not clear, it is a straightforward matter to show that the response of the system at point y to an external disturbance at point x is given by the $\langle i\hat{\phi}(x)\,\phi(y)\rangle$ propagator. We shall show how the elimination of this field combined with the results of Sections 3 and 4 leads to a more clearcut physical interpretation in terms of the fundamental equations governing the stochastic process.

The generating functional for which we wish to invoke BRS invariance is given by[10]

$$Z[J,\zeta,\bar{\zeta}]=\int D\phi\, D\bar{\eta}\, D\eta \exp\left\{-\frac{1}{2}(\partial_t\phi+F)^2 - \bar{\eta}\left(\partial_t+\frac{\delta F}{\delta\phi}\right)\eta+J\phi+\zeta\eta+\bar{\zeta}\bar{\eta}\right\} \tag{19}$$

This action is invariant under the transformation

$$\begin{aligned} \delta\eta &= 0 \\ \delta\bar{\eta} &= -\varepsilon(\partial_t\phi+F) \\ \delta\phi &= \varepsilon\eta \end{aligned} \tag{20}$$

with ε an infinitesimal anticommuting parameter as before, and where the result of integrating out the conjugate field leads to the replacement of $\hat{\phi}$

by the factor $i(\partial_t\phi + F)$. Note that (20) is not nilpotent, as opposed to (9). The Ward identity (11) now becomes

$$\left(J\frac{\delta}{\delta\zeta} + \bar{\zeta}\left(\partial_t\frac{\delta}{\delta J} + F\left[\frac{\delta}{\delta J}\right]\right)\right) Z[J, \zeta, \bar{\zeta}] = 0 \tag{21}$$

Proceeding with the analysis as in Section 4, one now finds that (14) is replaced by

$$\langle\bar{\eta}\eta\rangle = -\langle\phi(\partial_t\phi + F)\rangle \tag{22}$$

which clearly shows that the response of the system to an external disturbance is given by the $\bar{\eta}\eta$ propagator. This is not surprising, since, as noted before, the presence of the ghosts derives from a functional determinant in the path integral which arises from the system constraint imposed through the prescription of the external force. We feel that this physical interpretation of a constrained system is more directly accessible than that afforded by formalisms in which $\hat{\phi}$ appears but the ghosts do not.

Furthermore, it is easy to show that

$$\langle\partial_t\phi + F\rangle = 0 \tag{23}$$

as it must for the Gaussian white noise force, and that

$$\langle C_3[\eta, \phi]\eta\rangle = 0 \tag{24}$$

implying the vanishing of the $\eta\phi$ propagator. Similarly, from the path integral definition of the generating functional one may prove that the $\bar{\eta}\bar{\eta}$ and $\bar{\eta}\phi$ propagators must also vanish, leaving only the $\phi\phi$ and $\bar{\eta}\eta$ with non-vanishing propagators.

While the discussion to this point has been formal, it should be pointed out that these results have practical applications. In particular, the problem of calculating correlation functions of a turbulent incompressible fluid stirred by a white noise force can be described by a generating functional of the form (19), and the results of using the Ward identity (21) lead to considerable simplifications. This will be discussed more completely in a forthcoming article. It should also be noted that the possibility of eliminating the conjugate field is not restricted to the case of a white noise force.

6. CONCLUSIONS

The methods employed in this paper have allowed us to give a definitive answer to the question of the role of the determinant at the non-perturbative level. The analysis in Sections 1 and 4 should make it clear that, for practical calculations, the logical choice would be to drop the determinant together with the graphs containing $\phi\hat{\phi}$ loops, since that would result in a considerable simplification of the task at hand. To study the structure and properties of the theory in general, however, the ghost formulation offers a superior perspective in that it provides a linear realization of a symmetry that is difficult to see in the formalism outlined in Section 1. We have illustrated the use of this symmetry with examples that appear relatively simple in the light of the present methods, but require some computational effort, and cannot be solved in as general a fashion if one uses the older methods. These examples obviously do not exhaust the possible applications. In effect, the Ward identity (11) should prove most valuable in studies concerning the renormalizability of a particular model, since this invariance would have to be preserved in order to incorporate causality into the renormalized theory. The importance of a requirement of this kind is well known in QFT.

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